Optimalization of minimal revenue

M Kupsa

April 1, 2022

1 Theory

We would like to bet on a horse race where odds for every horse is offered. Assume, there are n horses with odds o_i, $i \leq n$. If you bet 1 dollar on the i-th horse (it is your wager you paid in advance), you will get $o_i$dollars if the i-th horse wins. To avoid technical issues, we assume $o_i > 0$ for every i (otherwise, the zero odds we simply ignore because they provide no profit)

Let us focus on a minimal revenue for a gambler that decomposes its 1 dollar into bets b_i, $i \leq n$, on particular horses. The revenue is a random variable $S(X) = o_X b_X$, where X is a random variable that express which horse win the race. A minimal revenue is given by the following formula:

$$W_{min} = \min_i b_i o_i.$$

It can be zero when a gambler use a strategie when at least one its bet b_i is zero. Bad luck will cost him all his bet. Denote

$$\beta_i = \frac{o_i^{-1}}{\sum_{i=1}^n o_i^{-1}}, \qquad i \leq n,$$

If he use these numbers that evidently sum up to one as the bets, namely $b_i = \beta_i$, $i \leq n$, then the risk-free revenue W_{min}satisfies:

$$W_{min} = \min_i \frac{o_i^{-1}}{\sum_{i=1}^n o_i^{-1}} \cdot o_i = \min_i \frac{1}{\sum_{i=1}^n o_i^{-1}} = \frac{1}{\sum_{i=1}^n o_i^{-1}}.$$

By a simple observation one can see that a minimal revenue is optimal for this bet. Generally,

$$\begin{aligned} W_{min} &= \min_i b_i o_i = \min_i (\beta_i o_i + (b_i - \beta_i) o_i) = \min_i \left(\frac{1}{\sum_{i=1}^n o_i^{-1}} + (b_i - \beta_i) o_i \right) \\ &= \frac{1}{\sum_{i=1}^n o_i^{-1}} + \min_i \left((b_i - \beta_i) o_i \right), \end{aligned}$$

where

$$\sum_{i=1}^{n}(b_i - \beta_i) = \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} \beta_i = 1 - 1 = 0.$$

If the sum of some terms is zero, then either they are all zeros, or some of the term is negative. In the first case, $b_i = \beta_i$, $i \leq n$, and the very last minimum in the equations is zero. Otherwise, $b_i - \beta_i < 0$ for some i, so the very last minimum is negative and the risk-free revenue is strictly smaller than the risk-free revenue for bets $b_i = \beta_i$, $i \leq n$.

Let us point out that we strongly used a fact that just only one possibility could happen. This led to the conclusion that any change from the optimal proportions β_i's gives worse risk-free part of the revenue. But if one can bet on possibilities that overlaps the optimal bets for risk-free revenue can change (in such a case some simplex method and linear programming is likely needed). Likewise, the optimal bets for expected value of revenue, when we do not ask for risk-free property is different. In such a case, we optimize a weigthed average of the odds instead of their minimum.

e.g. in a football match on any team, the remise and also the not only on 1,2

2 Example

2.1 Example 1

Let the football match has the following odds, Sparta wins 2, Banik wins 4, tie 5. It is definitely a super fair game, optimal bets are:

$$\beta_{Sparta} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4} + \frac{1}{5}} = \frac{\frac{1}{2}}{\frac{10+5+4}{20}} = \frac{10}{19}, \qquad \beta_{Banik} = \frac{5}{19}, \qquad \beta_{tie} = \frac{4}{19}.$$

The maximal risk-free revenue $W_{min} = \frac{20}{19}$ is attained with these bets. Hence, you will get one nineteenth of your wager for free without any risk.

2.2 Example 2

Another example has the following odds, Sparta wins 2, Banik wins 3, tie 4. It is definitely a unfair game, optimal bets are:

$$\beta_{Sparta} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{6+4+3}{12}} = \frac{6}{13}, \qquad \beta_{Banik} = \frac{4}{13}, \qquad \beta_{tie} = \frac{3}{13}.$$

The maximal risk-free revenue $W_{min} = \frac{12}{13}$ is attained with these bets. The revenue is smaller than one what means that it is a loss in fact. You will lose one thirteenth of your wager. But still, the bets above are optimal in the sense that with other bets you risk larger loss.

3 Log-optimal betting and bookmaker margin

The following formula is proved in Chapter 6 in Thomas and Cover book:

$$E(\log S(b)) := W(b) = D(p||b) - D(p||r),$$

where $r_i := o_i^{-1}$, p are real probabilities for winning of respective horses and b is the relative distribution of money of a gambler. Hence $\sum_i b_i = 1$, $\sum_i p_i = 1$. But the formula is not valid in all cases, but only when $\sum_i r_i = 1$. Since r_i's are interpreted as the estimates for p_i's by the bookmaker, the condition that the sum equals one can be interpreted that bookmaker has no margin on the race. It is quite unrealistic assumption. Nevertheless, the way to get general formula with good interpretation is easy. It is enough to normalize the sum of r_i's. This approach introduces the normalization constant into the formula and this normalization constant is already known measure with appropriate interpretation.

First, let us add to the whole picture the revenue for the bookmaker. This revenue is again *relative to the amount of money the gambler bets:*

$$T(b) = 1 - o_X b_X = 1 - S(b),$$

where X is the random variable that expresses which horse will win the race.

Let us leave change the definition of r_i's in the way, that they will be still proportional to o_i^{-1}, but normalized:

$$M = \frac{1}{\sum_i o_i^{-1}}, r_i = M \cdot o_i^{-1}, i \leq n \quad .$$

The normalization constant M has appeared in the previous section as the best revenue (over all our strategies) in the worst scenario over all possibilities of the race result. More literaly, M equals the maximal risk-free revenue W^*_{min}. From the bookmaker point of view, if r_i's are estimates for probabilies p_i's, to set $o_i := M \cdot r_i^{-1}$ with M independent from i, is a way how to establish a uniform margin on all possible events. Namely, under the assumption (the bookmaker likely do) $p_i = r_i$, a gambler and a bookmaker has constant expected revenue disrespect to the gambler strategy:

$$ES(b) = \textstyle\sum_i p_i o_i b_i = \sum M b_i = M, \quad ET(b) = E(1 - S(b)) = 1 - ES(b) = 1 - M \quad .$$

In realistic situation M is smaller than 1 and it represents the percentage of your bet that bookmaker aim to leave you, $1 - M$ is his margin (if he well estimated the probabilities). Nevertheless, the following formula works for whatever $M > 0$:

$$
\begin{aligned}
W(b) &= \sum_i p_i \log\left(o_i b_i\right) = \sum_i p_i \log\left(M r_i^{-1} b_i\right) \\
&= \sum_i p_i \left(\log M + \log\left(\frac{p_i}{r_i}\right) - \log\left(\frac{p_i}{b_i}\right)\right) \\
&= D(p||r) - D(p||b) + \log M.
\end{aligned}
$$

First two divergences measure the error in bookmaker's and gambler's estimates for p, respectively. The third element is connected to the bookmaker margin. If M is smaller than one, the bookmaker's margin $1 - M$ is positive and the term in the gambler's doubling rate is negative. The higher margin, the smaller gambler's doubling rate. If the margin is positive, the gambler's estimates must be better than the bookmaker's ones by $-\log M$. Otherwise, the doubling rate is negative and gambler is loosing money in long term.